

ON THE RANKIN–SELBERG ZETA-FUNCTION

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ABSTRACT. We obtain the approximate functional equation for the Rankin-Selberg zeta-function on the $1/2$ -line.

1. INTRODUCTION

Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2, \mathbb{Z})$, so that

$$\varphi\left(\frac{az+b}{cz+d}\right) = (cz+d)^\kappa \varphi(z) \quad (a, b, c, d \in \mathbb{Z}, ad - bc = 1)$$

when $\Im z > 0$ and $\lim_{\Im z \rightarrow \infty} \varphi(z) = 0$ (see e.g., R.A. Rankin [12] for basic notions). We denote by $a(n)$ the n -th Fourier coefficient of $\varphi(z)$ and suppose that $\varphi(z)$ is a normalized eigenfunction for the *Hecke operators* $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$ (see Rankin op. cit. for the definition and properties of the Hecke operators). The classical example is $a(n) = \tau(n)$, when $\kappa = 12$. This is the well-known *Ramanujan tau-function* defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x \left\{ (1-x)(1-x^2)(1-x^3) \cdots \right\}^{24} \quad (|x| < 1).$$

Let $c_n (\geq 0)$ be the convolution function defined by

$$(1.1) \quad c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} \left| a\left(\frac{n}{m^2}\right) \right|^2.$$

Note that c_n is a multiplicative arithmetic function, namely $c_{mn} = c_m c_n$ when $(m, n) = 1$, since $a(n)$ is multiplicative.

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The well-known *Rankin-Selberg problem* consists of the estimation of the error term function

$$(1.2) \quad \Delta(x) := \sum_{n \leq x} c_n - Cx.$$

The constant $C (> 0)$ in (1.2) may be written down explicitly (see e.g., [8]) as

$$C = C(\varphi) = \frac{2\pi^2(4\pi)^{\kappa-1}}{\Gamma(\kappa)} \iint_{\mathfrak{F}} y^{\kappa-2} |\varphi(z)|^2 dx dy,$$

the integral being taken over a fundamental domain \mathfrak{F} of the group $SL(2, \mathbb{Z})$. The classical upper bound for $\Delta(x)$ (strictly speaking $\Delta(x) = \Delta(x; \varphi)$) of Rankin and Selberg, obtained independently in their important works [11] and [14] of 1939, is

$$(1.3) \quad \Delta(x) = O(x^{3/5}).$$

In fact, this result is one of the longest standing unimproved bounds of Analytic Number Theory, but the present paper is not concerned with this problem. Our object of study is the so-called *Rankin-Selberg zeta-function*

$$(1.4) \quad Z(s) := \sum_{n=1}^{\infty} c_n n^{-s},$$

which is the generating *Dirichlet series* for the sequence $\{c_n\}_{n \geq 1}$. One can define the Rankin-Selberg zeta-function in various degrees of generality; see e.g., Li and Wu [10] where the authors establish universality properties of such functions.

Note that series in (1.4) converges absolutely for $\Re s > 1$. Namely from (1.2) and P. Deligne's estimate $|a(n)| \leq n^{(\kappa-1)/2} d(n)$ (see [1]), where $d(n) (\ll_{\varepsilon} n^{\varepsilon})$ is the number of positive divisors of n , we have

$$(1.5) \quad c_n \ll_{\varepsilon} n^{\varepsilon},$$

providing absolute convergence of $Z(s)$ for $\Re s > 1$.

Here and later ε denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while $a = O_{\varepsilon}(b)$ (same as $a \ll_{\varepsilon} b$) means that the constant implied by the O -symbol depends on ε .

For $\Re s \leq 1$ the function $Z(s)$ is defined by analytic continuation. It has a simple pole at $s = 1$ with residue C (cf. (1.1)), and is otherwise regular. For every $s \in \mathbb{C}$ it satisfies the functional equation

$$(1.6) \quad \Gamma(s + \kappa - 1) \Gamma(s) Z(s) = (2\pi)^{4s-2} \Gamma(\kappa - s) \Gamma(1 - s) Z(1 - s),$$

where $\Gamma(s)$ is the *gamma-function*. One has the decomposition

$$Z(s) = \zeta(2s) \sum_{n=1}^{\infty} |a(n)|^2 n^{1-\kappa-s},$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ ($\Re s > 1$) is the familiar *Riemann zeta-function*. This formula is the analytic equivalent of the arithmetic relation (1.1). In our context it is more important that one also has the decomposition

$$(1.7) \quad Z(s) := \sum_{n=1}^{\infty} c_n n^{-s} = \zeta(s) \sum_{n=1}^{\infty} b_n n^{-s} = \zeta(s) B(s),$$

say, where $B(s)$ belongs to the *Selberg class* of *Dirichlet series* of degree three. The coefficients b_n in (1.7) are multiplicative and satisfy

$$(1.8) \quad b_n \ll_{\varepsilon} n^{\varepsilon}.$$

This follows from

$$b_n = \sum_{d|n} \mu(d) c_{n/d},$$

which is a consequence of (1.7), the *Möbius inversion formula* and (1.5). Actually the coefficients b_n are bounded by a log-power (see [13]) in mean square, but this stronger property is not needed here. For the definition and basic properties of the Selberg class \mathcal{S} of L -functions the reader is referred to A. Selberg's seminal paper [15] and the comprehensive survey paper of Kaczorowski–Perelli [9].

In view of (1.8) the series for $B(s)$ converges absolutely for $\Re s > 1$, but $B(s)$ has analytic continuation which is holomorphic for $\Re s > 0$. This important fact follows from G. Shimura's work [16] (see also A. Sankaranarayanan [13]), and it implies that (1.7), namely $Z(s) = \zeta(s)B(s)$, holds for $\Re s > 0$ and not only for $\Re s > 1$. The function $B(s)$ is of degree three in \mathcal{S} , as its functional equation (see e.g., A. Sankaranarayanan [13]) is

$$B(s)\Delta_1(s) = B(1-s)\Delta_1(1-s),$$

$$\Delta_1(s) = \pi^{-3s/2} \Gamma(\tfrac{1}{2}(s + \kappa - 1)) \Gamma(\tfrac{1}{2}(s + \kappa)) \Gamma(\tfrac{1}{2}(s + \kappa + 1)).$$

It is very likely that $B(s)$ is primitive in \mathcal{S} , namely that it cannot be factored non-trivially as $F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$, but this seems hard to prove. Since $B(s)$ is holomorphic for $\Re s > 0$, it would follow that one of the factors, say $F_1(s)$, is $L(s + i\alpha, \chi)$ for some $\alpha \in \mathbb{R}$ and χ a primitive Dirichlet character. This follows from the fact that elements of degree one in \mathcal{S} are $\zeta(s + i\alpha)$ and $L(s + i\alpha, \chi)$. However, then $F_2(s)$ would have degree two in \mathcal{S} , but the classification of functions in \mathcal{S} of degree two is a difficult open problem.

2. THE APPROXIMATE FUNCTIONAL EQUATION FOR $Z(s)$

Approximate functional equations are an important tool in the study of Dirichlet series $F(s) = \sum_{n \geq 1} f(n)n^{-s}$. Their purpose is to approximate $F(s)$ by *Dirichlet polynomials* of the type $\sum_{n \leq x} f(n)n^{-s}$ in a certain region where the series defining $F(s)$ does not converge absolutely. In the case of the powers of $\zeta(s)$ they were studied e.g., in Chapter 4 of [5] and [6], and in a more general setting by the author [7].

Before we state our results, which involve approximations of $Z(s)$ by Dirichlet polynomials of the form $\sum_{n \leq x} c_n n^{-s}$, we need some notation. Let (see (1.6))

$$(2.1) \quad X(s) = \frac{Z(s)}{Z(1-s)} = (2\pi)^{4s-2} \frac{\Gamma(\kappa-s)\Gamma(1-s)}{\Gamma(s+\kappa-1)\Gamma(s)},$$

let $\tau = \tau(t)$ be defined by

$$(2.2) \quad \log \tau = -\frac{X'(\frac{1}{2} + it)}{X(\frac{1}{2} + it)} \quad (t \geq 3),$$

and

$$(2.3) \quad \Phi(w) = \Phi(w; s, \tau) := \tau^{w-s} X(w) - X(s) \quad (\tfrac{1}{2} \leq \sigma = \Re s \leq 1).$$

Then we have

THEOREM 1. *For $\frac{1}{2} \leq \sigma = \Re s \leq 1, t \geq 3, s = \sigma + it$ we have*

$$(2.4) \quad \begin{aligned} Z(s) &= \sum_{n \leq x} c_n n^{-s} + X(s) \sum_{n \leq y} c_n n^{s-1} + C_1 \frac{x^{1-s}}{1-s} + C_2 X(s) \frac{y^s}{s} \\ &+ O_\varepsilon \left\{ t^\varepsilon (x^{-\sigma} + hx^{1-\sigma}) + t^{2+\varepsilon-4\sigma} (y^{\sigma-1} + hy^\sigma) \right\} \\ &- \frac{1}{2\pi i h^3} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Z(1-z) \Phi(z; s, \tau) y^{s-z} (z-s)^{-4} \left(1 - e^{-h(s-z)}\right)^3 dz, \end{aligned}$$

where $xy = \tau, 1 \ll x, y \ll \tau, 0 < h \leq 1$ is a parameter to be suitably chosen, and C_1, C_2 are absolute constants.

The restriction $\frac{1}{2} \leq \sigma = \Re s \leq 1$ in Theorem 1 can be removed, and one can consider the whole range $0 \leq \sigma \leq 1$. For $0 \leq \sigma \leq \frac{1}{2}$ this is achieved on replacing s by $1-s$, interchanging x and y , and using $Z(1-s)X(s) = Z(s)$, together with (2.4) and (3.5) of Lemma 2.

The most important case of Theorem 1 is when $s = \frac{1}{2} + it$ lies on the so-called *critical line* $\Re s = \frac{1}{2}$. Then we obtain from (2.4) the following

THEOREM 2. For $s = \frac{1}{2} + it$, $t \geq 3$, $xy = \tau$, $1 \ll x, y \ll \tau$ we have

$$(2.5) \quad \begin{aligned} Z(s) = & \sum_{n \leq x} c_n n^{-s} + X(s) \sum_{n \leq y} c_n n^{s-1} + C_1 \frac{x^{1-s}}{1-s} + C_2 X(s) \frac{y^s}{s} \\ & + O_\varepsilon \left(t^{\varepsilon-11/16} (x^{1/2} + t^2 x^{-1/2})^{3/4} \right) + O_\varepsilon (t^{1/2+\mu(1/2)+\varepsilon}), \end{aligned}$$

where, for $\sigma \in \mathbb{R}$,

$$\mu(\sigma) := \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t}.$$

The best known result that $\mu(1/2) \leq 32/205 = 0.15609\dots$ is due to M.N. Huxley [4]. The famous *Lindelöf hypothesis* is that $\mu(1/2) = 0$ (equivalent to $\mu(\sigma) = 0$ for $\sigma \geq 1/2$), and it makes the second error term in (2.5) equal to $O_\varepsilon(t^{1/2+\varepsilon})$.

In general, if one introduces smooth weights in the sums in question, then the ensuing error terms are substantially improved. This was done e.g., in Chapters 4 of [5] and [6] and in [7]. From the Theorem of [7] (eqs. (19) and (20) with $\sigma = \frac{1}{2}$, $K = 4$, $t \geq 3$, $xy = \tau$, $1 \ll x, y \ll \tau$) we obtain

$$(2.6) \quad Z(s) = \sum_{n \leq x} \rho(n/x) c_n n^{-s} + X(s) \sum_{n \leq y} \rho(n/y) c_n n^{s-1} + O_\varepsilon(t^\varepsilon) \quad (s = \frac{1}{2} + it).$$

The smooth function $\rho(x)$ (see Chapter 4 of [6] for an explicit construction) is defined as follows. Let $b > 1$ be a fixed constant and $\rho(x) \in C^\infty(0, \infty)$,

$$\rho(x) + \rho(1/x) = 1 \quad (\forall x \in \mathbb{R}), \quad \rho(x) = 0 \quad (x \geq b).$$

There is another aspect of this subject worth mentioning. One can consider the function

$$(2.7) \quad \mathcal{Z}(t) := Z(\tfrac{1}{2} + it) X^{-1/2}(\tfrac{1}{2} + it) \quad (t \in \mathbb{R}).$$

The functional equation for $Z(s)$ in the form $Z(s) = X(s)Z(1-s)$ gives easily $X(s)X(1-s) = 1$, hence

$$\begin{aligned} \overline{\mathcal{Z}(t)} &= Z(\tfrac{1}{2} - it) X^{-1/2}(\tfrac{1}{2} - it) = Z(\tfrac{1}{2} + it) X(\tfrac{1}{2} - it) X^{-1/2}(\tfrac{1}{2} - it) \\ &= Z(\tfrac{1}{2} + it) X^{-1/2}(\tfrac{1}{2} + it) = \mathcal{Z}(t). \end{aligned}$$

Therefore $\mathcal{Z}(t) \in \mathbb{R}$ when $t \in \mathbb{R}$. The function $\mathcal{Z}(t)$ is the analogue of the classical *Hardy's function* $\zeta(\frac{1}{2} + it) \chi^{-1/2}(\frac{1}{2} + it)$, $\zeta(s) = \chi(s)\zeta(1-s)$, which plays a fundamental role in the study of the zeros of $\zeta(s)$ on the *critical line* $\Re s = 1/2$.

Taking $x = (t/2\pi)^2$ in Theorem 2, we obtain then with the aid of Lemma 2 the following

Corollary.

(2.8)

$$\mathcal{Z}(t) = 2 \sum_{n \leq (t/2\pi)^2} c_n n^{-1/2} \cos \left(t \log \left(\frac{(t/2\pi)^2}{n} \right) - 2t + (\kappa - 1)\pi \right) + O_\varepsilon(t^{\frac{1}{2} + \mu(\frac{1}{2}) + \varepsilon}).$$

One can compare (2.8) to the analogue for $Z^4(t) = |\zeta(\frac{1}{2} + it)|^4$, since (4.29) of [5] may be rewritten as

(2.9)

$$Z^4(t) = 2 \sum_{n \leq (t/2\pi)^2} d_4(n) n^{-1/2} \cos \left(t \log \left(\frac{(t/2\pi)^2}{n} \right) - 2t - \frac{1}{2}\pi \right) + O_\varepsilon(t^{13/48 + \varepsilon}),$$

where $d_4(n) = \sum_{abcd=n} 1$ is the divisor function generated by $\zeta^4(s)$. The reason why the error term in (2.9) is sharper than the one in (2.8) is because we have much more information on $\zeta^4(s)$ than on $Z(s)$.

The plan of the paper is as follows. In Section 3 we shall formulate and prove the lemmas necessary for the proofs. In Section 4 we shall prove Theorem 1, and in Section 5 we shall prove Theorem 2.

3. THE NECESSARY LEMMAS

Lemma 1. *We have*

$$(3.1) \quad \int_0^X |Z(\tfrac{1}{2} + it)| dt \ll_\varepsilon X^{5/4 + \varepsilon}.$$

Proof of Lemma 1. From the decomposition (1.7) and the Cauchy-Schwarz inequality for integrals we obtain

$$(3.2) \quad \int_{X/2}^X |Z(\tfrac{1}{2} + it)| dt \leq \left(\int_{X/2}^X |\zeta(\tfrac{1}{2} + it)|^2 dt \int_{X/2}^X |B(\tfrac{1}{2} + it)|^2 dt \right)^{1/2}.$$

Note that we have the elementary bound (see e.g., Chapter 1 of [5])

$$(3.3) \quad \int_0^X |\zeta(\tfrac{1}{2} + it)|^2 dt \ll X \log X,$$

and that $B(s)$ belongs to the Selberg class of degree three. Therefore $B(s)$ is analogous to $\zeta^3(s)$, and by following the proof of Theorem 4.4 of [5] (when $k = 3$)

it is seen that $B(s)$ satisfies an analogous approximate functional equation, with $M \geq (3X)^3/Y$, $X^\varepsilon \leq t \leq X$. Taking $Y = X^{3/2}$ and applying the *mean value theorem for Dirichlet polynomials* (Theorem 5.2 of [5]) we obtain, in view of (1.8), that

$$(3.4) \quad \int_{X/2}^X |B(\tfrac{1}{2} + it)|^2 dt \ll_\varepsilon X^{3/2+\varepsilon}.$$

The bound in (3.1) follows immediately from (3.2)–(3.4) if we replace X by $X/2^j$ ($j = 1, 2, \dots$) and add the resulting expressions. The best bound for the integral in (3.1) is, up to ‘ ε ’, $X^{1+\varepsilon}$. This follows e.g., by obvious modifications of the arguments used in the proof of Theorem 9.5 of [5]. It would improve the bound in (1.3) to $O_\varepsilon(x^{1/2+\varepsilon})$.

Lemma 2. *For $0 \leq \sigma \leq 1$ fixed, $t \geq 3$, we have*

$$(3.5) \quad X(\sigma + it) = \left(\frac{t}{2\pi}\right)^{2-4\sigma} \exp\left(4it - 4it \log\left(\frac{t}{2\pi}\right) + (1 - \kappa)\pi i\right) \cdot \left(1 + O\left(\frac{1}{t}\right)\right),$$

where the O -term admits an asymptotic expansion in negative powers of t .

Proof of Lemma 2. Follows from (2.1) and the full form of Stirling’s formula, namely

$$\log \Gamma(s + b) = (s + b - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log 2\pi + \sum_{j=1}^K \frac{(-1)^j B_{j+1}(b)}{j(j+1)s^j} + O_\delta\left(\frac{1}{|s|^{K+1}}\right),$$

which is valid for b a constant, any fixed integer $K \geq 1$, $|\arg s| \leq \pi - \delta$ for $\delta > 0$, where the points $s = 0$ and the neighbourhoods of the poles of $\Gamma(s + b)$ are excluded, and the $B_j(b)$ ’s are *Bernoulli polynomials*; for this see e.g., A. Erdélyi et al. [2].

Lemma 3. *Let $\tau = \tau(t)$ be defined by (2.2). Then*

$$(3.6) \quad \tau = \left(\frac{t}{2\pi}\right)^4 \left(1 + O\left(\frac{1}{t^2}\right)\right) \quad (t \geq 3),$$

where the O -term admits an asymptotic expansion in negative powers of t . If $\Phi(w)$ is defined by (2.3), then $\Phi(w)(s - w)^{-2}$ is regular for $\Re w \leq \frac{1}{2}$ and also for $\Re w < \sigma$ if $\frac{1}{2} < \sigma \leq 1$. Moreover, uniformly in s for $\Re w = \frac{1}{2}$, $t \geq 3$ we have

$$(3.7) \quad \Phi(w) \ll t^{2-4\sigma} \min\left\{1, \min\left(t^{-1}|w - s|^2\right)\right\}.$$

Proof of Lemma 3. The functions τ and Φ were introduced, in the case of $\zeta^2(s)$, by Hardy and Littlewood in their classical proof [3] of the approximate functional equation for $\zeta^2(s)$. To prove (3.6) recall (see (2.1)) that

$$X(s) = \frac{Z(s)}{Z(1-s)} = (2\pi)^{4s-2} \frac{\Gamma(\kappa-s)\Gamma(1-s)}{\Gamma(s+\kappa-1)\Gamma(s)}.$$

Logarithmic differentiation gives then

$$\begin{aligned} -\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)} &= -4\log(2\pi) + \frac{\Gamma'(\kappa-\frac{1}{2}-it)}{\Gamma(\kappa-\frac{1}{2}-it)} + \frac{\Gamma'(\frac{1}{2}-it)}{\Gamma(\frac{1}{2}-it)} \\ &\quad + \frac{\Gamma'(\kappa-\frac{1}{2}+it)}{\Gamma(\kappa-\frac{1}{2}+it)} + \frac{\Gamma'(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}+it)}. \end{aligned}$$

If we use (see (A.35) of [5])

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right) \quad (|\arg s| \leq \pi - \delta, |s| \geq \delta),$$

where the O -term has an asymptotic expansion in term of negative powers of s , we obtain

$$\log \tau = -\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)} = 4\log t - 4\log(2\pi) + O\left(\frac{1}{t^2}\right) \quad (t \geq 3),$$

which is equivalent to (3.6).

The only non-trivial case concerning the regularity of $\Phi(w)(s-w)^{-2}$ is when $w = \frac{1}{2} + iv$, $s = \frac{1}{2} + it$, and this follows from (3.7). For $w = \frac{1}{2} + iv$ we have

$$|\Phi(w)| \leq \tau^{1/2-\sigma} |X(\frac{1}{2}+iv)| + |X(\sigma+it)| \ll t^{2-4\sigma}$$

in view of (3.6) and (3.5).

To obtain the other bound in (3.7) suppose that $|w-s| \ll \sqrt{t}$, which is the relevant range of its validity. Then, for $w = \frac{1}{2} + iv$, we have $v \asymp t$ and

$$\frac{d^2}{dw^2} X(w) \asymp \frac{1}{t} \quad (w = \frac{1}{2} + iv, v \asymp t).$$

Write (2.3) as

$$(3.8) \quad \Phi(w) = \tau^{w-s} X(w) \left(1 - \frac{X(s)}{X(w)} \tau^{s-w}\right)$$

and note that, by Taylor's formula,

$$\begin{aligned}
\frac{X(s)}{X(w)}\tau^{s-w} &= \exp\left(\log X(s) - \log X(w) + (s-w)\log \tau\right) \\
&= \exp\left((s-w)\frac{X'(w)}{X(w)} + O(|s-w|^2t^{-1}) + (s-w)\log \tau\right) \\
&= \exp\left((s-w)\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)} + O(|s-w|^2t^{-1}) + (s-w)\log \tau\right) \\
&= 1 + O(|s-w|^2t^{-1}),
\end{aligned}$$

in view of (2.3). If we insert this in (3.8) we obtain the second estimate in (3.7) from (3.5), (3.6) and (3.8).

4. PROOF OF THEOREM 1

The idea of proof of Theorem 1 goes back to Hardy-Littlewood [3], who considered the approximate functional equation for $\zeta^2(s)$. R. Wiebelitz [17] generalized their method to deal with $\zeta^k(s)$ when $k \in \mathbb{N}, k > 2$, and this was refined in Theorem 4.3 of [5]. In what follows we shall make the modifications which are necessary in the case of $Z(s)$. Let the hypotheses of Theorem 1 hold and set

$$\begin{aligned}
I = I(s, x) &:= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Z(s+w)x^w w^{-4} dw \\
&= \sum_{n=1}^{\infty} c_n n^{-s} \left\{ \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{x}{n}\right)^w w^{-4} dw \right\} \\
&= \frac{1}{3!} \sum_{n \leq x} c_n n^{-s} \log^3(x/n) := S_x,
\end{aligned}$$

say, where we used the absolute convergence of $Z(s)$ for $\sigma > 1$ and (A.12) of [5] with $k = 4$ (reflecting the fact that $Z(s)$ belongs to the Selberg class of degree $k = 4$). The basic idea is to use a differencing argument to recover $\sum_{n \leq x} c_n n^{-s}$ from the same sum weighted by $\log^3(x/n)$. To achieve this, first we move the line of integration in I to $\Re w = -1/4$. In doing this we pass over the poles $w = 0$ and $w = 1 - s$ of the integrand, with the respective residues

$$F_x := \sum_{m=0}^3 \frac{Z^{(m)}(s)}{m!(3-m)!} (\log x)^{3-m}$$

and

$$Q_x := \frac{Cx^{1-s}}{(1-s)^4}.$$

Hence by the residue theorem we obtain

$$(4.1) \quad J_0 := \frac{1}{2\pi i} \int_{-1/4-i\infty}^{-1/4+i\infty} Z(s+w)x^w w^{-4} dw = I - F_x - Q_x = S_x - F_x - Q_x.$$

In the integral in (4.1) set $z = s + w$, replace x by τ/y , and use the functional equation for $Z(s)$ and (2.3) in the form

$$\tau^{u-s} X(u) = X(s) + \Phi(u; s, \tau),$$

to obtain

$$\begin{aligned} J_0 &= \frac{1}{2\pi i} \int_{-1/4-i\infty}^{-1/4+i\infty} Z(1-z)X(s)y^{s-z}(z-s)^{-4} dz \\ &\quad + \frac{1}{2\pi i} \int_{-1/4-i\infty}^{-1/4+i\infty} Z(1-z)\Phi(z; s, \tau)y^{s-z}(z-s)^{-4} dz \\ &= X(s)J_1 + J_2, \end{aligned}$$

say. This is the point which explains the definition of the function Φ in (2.3). We use again (A.12) of [5] to deduce that

$$J_1 = \frac{1}{3!} \sum_{n \leq y} c_n n^{s-1} \log^3(x/n) := S_y,$$

similarly to the notation used in evaluating I . The line of integration in J_2 is moved to $\Re z = 1/4$. We pass over the pole $z = 0$ of the integrand, picking the residue which is

$$Q_y := -\frac{C y^s}{s^4}.$$

Therefore from (4.1) we obtain

$$(4.2) \quad F_x - S_x + Q_x = -X(s)(S_y - Q_y) - J_y$$

with

$$J_y := \frac{1}{2\pi i} \int_{1/4-i\infty}^{1/4+i\infty} Z(1-z)\Phi(z; s, \tau)y^{s-z}(z-s)^{-4} dz.$$

In (4.2) we replace x and y by $x e^{\nu h}$ and $y e^{-\nu h}$ ($0 \leq \nu \leq 3$), respectively, so that the condition $x e^{\nu h} \cdot y e^{-\nu h} = \tau$ is preserved. We use (see (4.39) and (4.40) of [5])

$$(4.3) \quad \sum_{\nu=0}^m (-1)^\nu \binom{m}{\nu} \nu^p = m! \quad (p \in \mathbb{N})$$

when $p = m$, and that the sum equals zero when $p < m$, and the estimate

$$e^z = \sum_{n=0}^M \frac{z^n}{n!} + O(|z|^{M+1}) \quad (M \geq 1, a \leq \Re z \leq b),$$

where a and b are fixed. To distinguish better the sums which will arise in this process we introduce left indices to obtain from (4.2)

$$\sum_{\nu=0}^3 (-1)^\nu \binom{3}{\nu} \left({}_\nu F_x - {}_\nu S_x + {}_\nu Q_x + X(s)({}_\nu S_y - {}_\nu Q_y) + {}_\nu J_y \right) = 0,$$

or abbreviating,

$$(4.4) \quad \bar{F}_x - \bar{S}_x + \bar{Q}_x + X(s)\bar{S}_y - X(s)\bar{Q}_y + \bar{J}_y = 0.$$

Each term in (4.4) will be evaluated or estimated separately. We have

$$\bar{F}_x = \sum_{m=0}^3 \frac{Z^{(m)}(s)}{3!(3-m)!} A_m(x)$$

with

$$\begin{aligned} A_m(x) &:= \sum_{\nu=0}^3 (-1)^\nu \binom{3}{\nu} (\log x + \nu h)^{3-m} \\ &= \sum_{r=0}^{3-m} \binom{3-m}{r} h^r \log^{3-m-r} x \sum_{\nu=0}^3 (-1)^\nu \binom{3}{\nu} \nu^r = 3! h^3 \end{aligned}$$

for $m = 0$, and otherwise $A_m(x) = 0$, where we used (4.3). Therefore

$$\bar{F}_x = h^3 Z(s),$$

and this is exactly what is needed for the approximate functional equation that will follow on dividing (4.4) by h^3 . Consider next

$$\begin{aligned} \bar{S}_x &= \frac{1}{3!} \sum_{n \leq x} c_n n^{-s} \sum_{\nu=0}^3 \binom{3}{\nu} (-1)^\nu \left(\nu h + \log(x/n) \right)^3 \\ &\quad + \frac{1}{3!} \sum_{\nu=0}^3 \binom{3}{\nu} (-1)^\nu \sum_{x < n \leq x e^{\nu h}} c_n n^{-s} \left(\nu h + \log(x/n) \right)^3 \\ &= \sum_1 + \sum_2, \end{aligned}$$

say. Analogously to the evaluation of \bar{F}_x it follows that

$$\sum_1 = h^3 \sum_{n \leq x} c_n n^{-s}.$$

We estimate \sum_2 trivially, on using (1.5), to obtain

$$\begin{aligned} \left| \sum_2 \right| &\leq \frac{1}{3!} \sum_{\nu=0}^3 \binom{3}{\nu} (2\nu h)^3 x^{-\sigma} \sum_{x < n \leq x e^{3h}} c_n \\ &\ll_{\varepsilon} h^3 x^{-\sigma} t^{\varepsilon} (1 + x(e^{3h} - 1)) \ll_{\varepsilon} t^{\varepsilon} (h^3 x^{-\sigma} + h^4 x^{1-\sigma}). \end{aligned}$$

In a similar way it follows that

$$\begin{aligned} -X(s) \bar{S}_y &= h^3 X(s) \sum_{n \leq y} c_n n^{s-1} + O_{\varepsilon} \left(h^3 |X(\sigma + it)| \sum_{\nu=0}^3 \sum_{y e^{-3h} < n \leq y} c_n n^{\sigma-1} \right) \\ &= h^3 X(s) \sum_{n \leq y} c_n n^{s-1} + O_{\varepsilon} \left(h^3 t^{2+\varepsilon-4\sigma} (y^{\sigma-1} + h y^{\sigma}) \right). \end{aligned}$$

Also

$$\bar{Q}_x = 3! h^3 C \frac{x^{1-s}}{1-s} + O(h^4 x^{1-\sigma}), \quad X(s) \bar{Q}_y = C_2 X(s) h^3 \frac{y^s}{s} + O_{\varepsilon} \left(t^{2+\varepsilon-4\sigma} h^4 y^{\sigma} \right).$$

Therefore we are left with the evaluation of

$$\bar{J}_y = \frac{1}{2\pi i} \int_{1/4-i\infty}^{1/4+i\infty} Z(1-z) \Phi(z; s, \tau) y^{s-z} (z-s)^{-4} \sum_{\nu=0}^3 (-1)^{\nu} \binom{3}{\nu} e^{-\nu h(s-z)} dz.$$

Observing that (3.7) holds and that the function

$$\sum_{\nu=0}^3 (-1)^{\nu} \binom{3}{\nu} e^{-\nu h(s-z)} = \left(1 - e^{-h(s-z)} \right)^3$$

has a zero of order three at $z = s$, we can move the line of integration in \bar{J}_y to $\Re z = \frac{1}{2}$. Hence

$$\bar{J}_y = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} Z(1-z) \Phi(z; s, \tau) y^{s-z} (z-s)^{-4} \left(1 - e^{-h(s-z)} \right)^3 dz.$$

Therefore we obtain the assertion of Theorem 1 from (4.4) if we divide the whole expression by h^3 and collect the above estimates for the error terms.

5. PROOF OF THEOREM 2

We set $s = \frac{1}{2} + it$, $z = \frac{1}{2} + iv$ in (2.4) and write the integral on the right-hand side as

$$(5.1) \quad i \int_{-\infty}^{\infty} \cdots dv = i \left(\int_{-\infty}^{t/2} + \int_{t/2}^{2t} + \int_{2t}^{\infty} \right) \cdots dv = i(I_1 + I_2 + I_3),$$

say. The integrals I_1 and I_3 are estimated analogously. The latter is, by trivial estimation and the first bound in (3.7),

$$(5.2) \quad \begin{aligned} & \int_{2t}^{\infty} Z(\tfrac{1}{2} - iv) \Phi(\tfrac{1}{2} + iv; s, \tau) y^{i(t-v)} (t-v)^{-4} \left(1 - e^{-hi(t-v)}\right)^3 dv \\ & \ll \int_{2t}^{\infty} |Z(\tfrac{1}{2} + iv)| v^{-4} dv \ll_{\varepsilon} t^{\varepsilon-11/4}, \end{aligned}$$

where we used (3.1) of Lemma 1. From (2.4), (5.1) and (5.2) it follows that

$$(5.3) \quad \begin{aligned} Z(s) &= \sum_{n \leq x} c_n n^{-s} + X(s) \sum_{n \leq y} c_n n^{s-1} + C_1 \frac{x^{1-s}}{1-s} + C_2 X(s) \frac{y^s}{s} \\ &+ O_{\varepsilon} \left(1 + t^{\varepsilon-11/16} (x^{1/2} + t^2 x^{-1/2})^{3/4} \right) - \frac{1}{2\pi i h^3} I_2, \end{aligned}$$

with the choice

$$h = t^{-11/16} (x^{1/2} + t^2 x^{-1/2})^{-1/4},$$

so that $0 < h \leq 1$ holds. To estimate I_2 we use

$$\left(1 - e^{-hi(t-v)}\right)^3 \ll h^3 |t-v|^3$$

and the second bound in (3.7) ($\sigma = \frac{1}{2}$). This gives, on using the Cauchy-Schwarz inequality for integrals,

$$(5.4) \quad \begin{aligned} h^{-3} I_2 &\ll \int_{t/2}^{2t} |Z(\tfrac{1}{2} + iv)| \min \left(\frac{1}{|t-v|}, \frac{|t-v|}{v} \right) dv \\ &\ll \left(\int_{t/2}^{2t} |Z(\tfrac{1}{2} + iv)|^2 dv \right)^{1/2} (j_1 + j_2 + j_3)^{1/2}, \end{aligned}$$

say. By (1.7), (3.4) and the definition of the μ -function we have

$$(5.5) \quad \int_{t/2}^{2t} |Z(\tfrac{1}{2} + iv)|^2 dv = \int_{t/2}^{2t} |\zeta(\tfrac{1}{2} + iv)|^2 |B(\tfrac{1}{2} + iv)|^2 dv \ll_{\varepsilon} t^{2\mu(1/2)+3/2+\varepsilon}.$$

We have

$$j_1 := \int_{t/2}^{t-\sqrt{t}} \frac{dv}{(t-v)^2} \ll \frac{1}{\sqrt{t}},$$

and the same bound holds for

$$j_3 := \int_{t+\sqrt{t}}^{2t} \frac{dv}{(t-v)^2}.$$

We also have

$$j_2 := \int_{t-\sqrt{t}}^{t+\sqrt{t}} (t-v)^2 \frac{dv}{v^2} \ll \frac{1}{\sqrt{t}},$$

so that from (5.4), (5.5) and the bounds for j_1, j_2, j_3 we infer that

$$(5.6) \quad h^{-3} I_2 \ll_{\varepsilon} t^{1/2+\mu(1/2)+\varepsilon}.$$

The assertion of Theorem 2 follows then from (5.3) and (5.6), since the first error term in (5.3) is absorbed by the right-hand side of (5.6) because $x^{1/2} \ll t^2$.

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